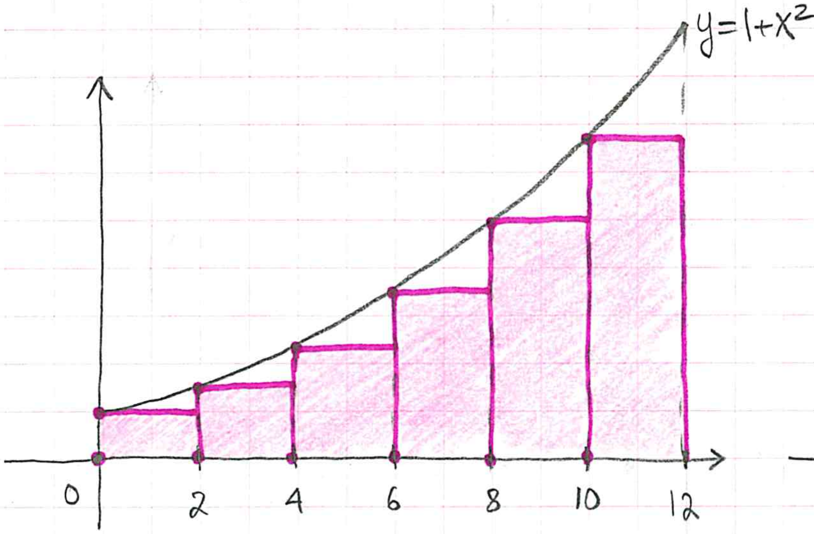
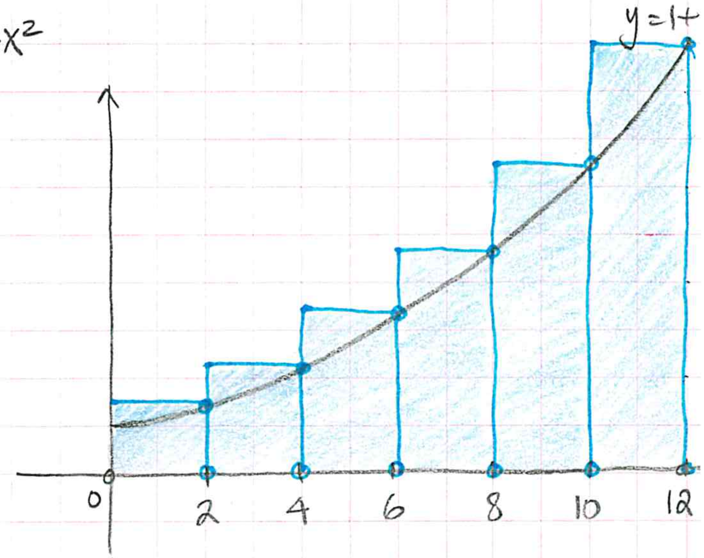
4.1. The Area Problem

Goal: Find the area under a curve $y=f(x)$ between $x=a$ and $x=b$.

Method: Approximate area by using rectangles.

Example: $y=1+x^2$; area under the curve b/w $x=0$ and $x=12$?





L_6 = sum obtained from 6 rectangles, using the left endpoint:

$$L_6 = 2\cdot f(0) + 2\cdot f(2) + 2\cdot f(4) + $$
$$+ 2\cdot f(6) + 2\cdot f(8) + 2\cdot f(10)$$
$$= 2\cdot(1+0^2) + 2\cdot(1+2^2) + 2\cdot(1+4^2) +$$
$$+2\cdot(1+6^2) + 2\cdot(1+8^2) + 2\cdot(1+10^2)$$
$$= 425$$

R_6 = sum obtained from 6 rectangles, using the right endpoint.

$$R_6 = 2\cdot f(2) + 2\cdot f(4) + 2\cdot f(6)$$
$$+ 2\cdot f(8) + 2\cdot f(10) + 2\cdot f(12)$$
$$= 740$$

⇒ The area under $y=1+x^2$, b/w $x=0$ & $x=12$ is b/w 425 & 740

⇒ How to improve accuracy? Increase the number of rectangles!

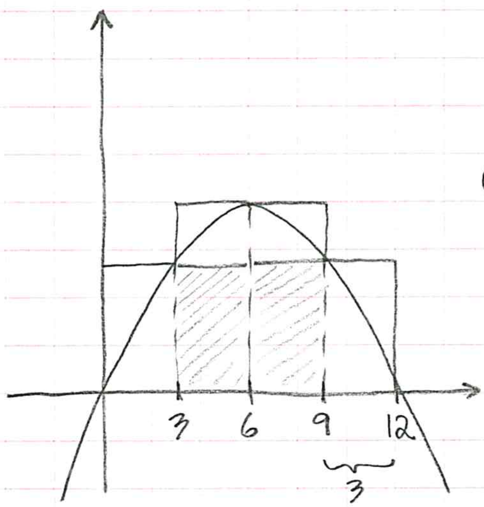
⇒ Building up to:

$$A = \int_0^{12} (1+x^2)\,dx \underset{\text{FTC}}{=} \left(x+\frac{x^3}{3}\right)\Big|_0^{12} = \left(12+\frac{12^3}{3}\right) - \left(0+\frac{0^3}{3}\right)$$

$$= 588 \text{ (The exact area!)}$$

Example: Area under $y = 12x - x^2$ from $x=0$ to $x=12$:
Use 4 rectangles and find: (a) an overestimate (upper sum)
(b) an underestimate (lower sum)

$y = 12x - x^2 = x(12-x) \Rightarrow$ zeroes at $x=0$, $x=12$; vertex at $\frac{-12}{-2} = 6$.



(a) overestimate:

$3 \cdot f(3) + 3 \cdot f(6) + 3 \cdot f(6) + 3 \cdot f(9) = 378$

(b) underestimate:

$3 \cdot f(0) + 3 \cdot f(3) + 3 \cdot f(9) + 3 \cdot f(12) = 162$

The Sigma Notation:

$$\sum_{i=m}^{n} f(i) \quad \text{or} \quad \sum_{i=m}^{n} a_i$$

n tells us to stop at $i=n$

$i=m$ tells us to start at $i=m$

$f(i)$ or a_i tell us what to add

Example:

$$\sum_{i=4}^{9} i = 4+5+6+7+8+9 = 39$$

$$\sum_{i=4}^{9} i^2 = 4^2+5^2+6^2+7^2+8^2+9^2 = 271$$

$$\sum_{i=4}^{9} (-1)^{i+1}(3i) = \underbrace{(-1)^5(3\cdot4)}_{n=4} + \underbrace{(-1)^6(3\cdot5)}_{n=5} + \underbrace{(-1)^7(3\cdot6)}_{n=6}$$

$$+ \underbrace{(-1)^8(3\cdot7)}_{n=7} + \underbrace{(-1)^9(3\cdot8)}_{n=8} + \underbrace{(-1)^{10}(3\cdot9)}_{n=9}$$

$$= -12+15-18+21-24+27 = 9$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} \qquad 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

Proof by Induction:

Statement S_n: $\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$.

I. S_1 is true: $\sum_{i=1}^{1} i = 1$; $\frac{1(1+1)}{2} = 1$. ✓

II. If S_n is true, then S_{n+1} is true:

Suppose it is <u>true</u> that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Then $\sum_{i=1}^{n+1} i = 1+2+\ldots+n+(n+1) = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1)$

$$= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

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Example: $\sum_{i=1}^{117} (8i+6) = 8 \cdot \left(\sum_{i=1}^{117} i\right) + \left(\sum_{i=1}^{117} 6\right)$

$$= 8 \cdot \frac{117 \cdot 118}{2} + 117 \cdot 6 = 117(4 \cdot 118 + 6).$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

Example: $\sum_{i=1}^{n} (9i^2+8i) = 9 \cdot \left(\sum_{i=1}^{n} i^2\right) + 8 \cdot \left(\sum_{i=1}^{n} i\right) = 9 \cdot \frac{n(n+1)(2n+1)}{6} + 8 \cdot \frac{n(n+1)}{2}$

Example: $\sum_{i=9}^{n} (7i) = \sum_{i=1}^{n} (7i) - \sum_{i=1}^{8} (7i) = 7 \cdot \frac{n(n+1)}{2} - 7 \cdot \frac{8 \cdot 9}{2}$